Properties of a Certain Projectively Defined Two-Parameter Family of Curves on a General Surface.

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1. *Analytic Foundation of the Differential Geometry of Non-Ruled Surfaces.*

In his first memoir on the "Projective Differential Geometry of Curved Surfaces,"* Mr. Wilczynski has shown that the projective theory of non-ruled analytic surfaces may be based on the consideration of a system of completely integrable partial differential equations of the second order which may be reduced to the so-called canonical form

$$y_{uu}+2by_v+fy=0, \quad y_{vv}+2a'y_u+gy=0, \tag{1}$$

where the subscripts denote partial differentiation, and where the coefficients are analytic functions of u and v satisfying the integrability conditions

$$\left.\begin{array}{l} a'_{uu}+g_u+2ba'_v+4a'b_v=0, \quad b_{vv}+f_v+2a'b_u+4ba'_u=0, \\ g_{uu}-f_{vv}-4fa'_u-2a'f_u+4gb_v+2bg_v=0. \end{array}\right\} \tag{2}$$

Such a system of differential equations possesses exactly four linearly independent analytic solutions

$$y^{(k)}=f^{(k)}(u, v) \qquad (k=1, 2, 3, 4). \tag{3}$$

If we now interpret $y^{(1)}, \ldots, y^{(4)}$, as the homogeneous coordinates of a point P_y, and let the independent variables range over all their values, P_y will generate a surface S_y, an integral surface of (1), which will be a ruled surface if, and only if, $a'=0$ or $b=0$ (a case which we shall exclude in this paper), and upon which the reference curves, $u=$constant and $v=$constant, are the

* There are five of these memoirs which appeared in the *Transactions of the American Mathematical Society* from 1907–1909. These will be referred to in the following pages as "First Memoir," etc.

asymptotic lines. Since the most general system of linearly independent solutions of (1) is of the form

$$\eta_i = \sum_{k=1}^{4} c_{ik} y^{(k)} \qquad (i=1, 2, 3, 4), \tag{4}$$

where $|c_{ik}| \neq 0$, the most general integrating surface of (1) is a projective transformation of any particular one.*

The canonical form is not uniquely determined. The most general transformation leaving it invariant will preserve the asymptotic curves as lines of reference and will be of the form

$$\bar{y} = C\sqrt{\alpha_u \beta_v}\, y, \quad \bar{u} = \alpha(u), \quad \bar{v} = \beta(v), \tag{5}$$

where α and β are arbitrary functions of u alone and of v alone respectively, and where C is an arbitrary constant.†

The functions

$$y = y, \quad y_u = z, \quad y_v = \rho, \quad y_{uv} = \sigma \tag{6}$$

are semicovariants. If the four independent solutions of (1) are substituted in z we get four functions $z^{(1)}, \ldots, z^{(4)}$, which may be taken as the homogeneous coordinates of a point P_z. So also for ρ and σ. The points P_y, P_z, P_ρ, P_σ are in general the vertices of a non-degenerate semicovariant tetrahedron T.‡ In just the same way every expression of the form

$$x = x_1 y + x_2 z + x_3 \rho + x_4 \sigma \tag{7}$$

determines a point P_x whose coordinates referred to T, by means of a suitable choice of the unit point, may be taken as (x_1, x_2, x_3, x_4).

2. *The Differential Equation of Certain Two-Parameter Families of Curves on a General Surface.*

The theory of two-parameter families of curves on a general surface has received but little attention except in so far as such a general theory may be implied by the theory of geodesics. We shall discuss in this paper a class of curves which will include the geodesics as a special case.

Let us associate with every point P_y of the surface one of the lines l_y which passes through that point, but does not lie in the tangent plane of the point. All these lines form a congruence L. Let us consider a curve on the surface which has the property that each of its osculating planes passes through the corresponding line of the congruence. All such curves will clearly

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* First Memoir, p. 237.   † First Memoir, pp. 90–95.   ‡ Second Memoir, pp. 79–80.

form a two-parameter family, and it is easy to show that they will be the integral curves of an equation of the form

$$u''v'-u'v''+2(bu'^3-a'v'^3)+2(p_1u'^2v'+p_2u'v'^2)=0, \tag{8}$$

where $u'=du/dt$, $u''=d^2u/dt^2$, etc., and where p_1 and p_2 are functions of u and v which depend upon the choice of the congruence L.

The coordinates of P_y referred to the local tetrahedron of reference T are $(1, 0, 0, 0)$ so that the equation of any line l_y through P_y referred to T may be written in the form

$$A_2x_2+A_3x_3+A_4x_4=0, \quad B_2x_2+B_3x_3+B_4x_4=0, \tag{9}$$

where the coefficients are in general functions of u and v. If we denote by Π_1 and Π_2 the left-hand members of (9),

$$\lambda\Pi_1+\mu\Pi_2=0 \tag{10}$$

is the equation of any plane through l_y. If, in particular, this plane is the osculating plane of a surface curve through P_y, the coordinates of y' and y'' must also satisfy (10). By means of (6),

$$\left.\begin{aligned} y' &= dy/dt=u'z+v'\rho, \\ y'' &= d^2y/dt^2=-(u'^2f+v'^2g)y+(u''-2a'v'^2)z+(v''-2bu'^2)\rho+2u'v'\sigma. \end{aligned}\right\} \tag{11}$$

The coordinates of y' and y'' referred to T are therefore

$$(0, u', v', 0), \quad (-u'^2f-v'^2g, \ u''-2a'v'^2, \ v''-2bu'^2, \ 2u'v').$$

The substitution of these coordinates in (10) gives upon the elimination of the ratio $\lambda:\mu$ equation (8), where

$$p_1=\frac{A_2B_4-A_4B_2}{A_3B_2-A_2B_3}, \quad p_2=\frac{A_3B_4-A_4B_3}{A_3B_2-A_2B_3}.$$

It is now evident that the line l_y must not lie in the tangent plane, as the denominator $A_3B_2-A_2B_3$ would vanish in that case and only in that case. If v be taken as the parameter of the curve, equation (8) will become

$$u''+2bu'^3+2p_1u'^2+2p_2u'-2a'=0. \tag{12}$$

It is also true that the one-parameter families of planes osculating the integral curves of an equation of the form (8) at the point P_y form a pencil. The equation of such a plane is

$$v'x_2-u'x_3+(p_1u'+p_2v')x_4=0. \tag{13}$$

If (u_1', v_1'), (u_2', v_2'), (u_3', v_3') are three pairs of values of u', v' corresponding to three distinct curves of this sort which pass through the same point, it is

evident that the corresponding equations (13) are not linearly independent for the determinant of the coefficients

$$\begin{vmatrix} v_1' & -u_1' & p_1u_1'+p_2v_1' \\ v_2' & -u_2' & p_1u_2'+p_2v_2' \\ v_3' & -u_3' & p_1u_3'+p_2v_3' \end{vmatrix}$$

vanishes identically. We have proved the following theorem:

If a two-parameter family of curves on a non-ruled surface has the property that the osculating planes of all of the curves of the family which pass through a given surface point P_y have in common a line through P_y, then the second order differential equation of the two-parameter family of curves has the form (8) *and conversely.*

These curves we have called *union curves* because of the characteristic property of united position of line and plane. It is evident that neither of the one-parameter families of asymptotic curves u=constant and v=constant can be union curves on a non-ruled surface since that would necessitate the condition $a'=0$, or $b=0$.

Among the congruences associated with the surface in the way described above are two of particular interest, the congruence of surface normals and the congruence of directrices of the second kind. One may define a geodesic as the curve whose osculating plane at a point contains the surface normal for that point. In case that for a particular one of the projectively equivalent integral surfaces of (1), the congruence L happens to be the congruence of surface normals, the corresponding union curves on that surface will be geodesics. Since perpendicularity is not a projective property, they would not in general be geodesics on the other integral surfaces of (1). The other congruence, the congruence of directrices of the second kind, is determined as follows: If we take five consecutive tangents to the curve v=constant, they determine in general a linear complex which approaches a limiting position as the tangents approach coincidence.* This complex is called the *osculating linear complex* for the asymptotic curve of the first kind. Similarly for the asymptotic curve of the second kind u=constant.† These complexes have as their intersection a congruence, one of whose directrices lies in the tangent plane of the point P_y, while the other, known as *directrix of the second kind,*

* E. J. Wilczynski, "Projective Differential Geometry," p. 162. In the following pages this book will be referred to as Proj. Diff. Geom.

† Second Memoir, pp. 90–95.

passes through the point itself, but does not lie in the tangent plane. The equations of the latter referred to the local tetrahedron T are

$$2bx_2+b_vx_4=0, \quad 2a'x_3+a'_ux_4=0. \tag{14}$$

If the congruence L is the directrix congruence of the second kind,

$$p_1=-\alpha, \quad p_2=\beta, \tag{15}$$

where

$$\alpha=a'_u/2a', \quad \beta=b_v/2b. \tag{16}$$

Equation (12) is a non-linear, non-homogeneous equation of the second order whose coefficients are functions of u and v. Since the complete solution involves two arbitrary constants, it represents a two-parameter family of curves, any one of which is uniquely determined when the surface point through which it must pass and the tangent at the point are given. The differential equation (12) can be integrated only in a few cases. The integration of an equation of this type in certain instances has been discussed by Darboux,* Liouville,† and Guldberg.‡ If L is the directrix congruence we can find a class of surfaces for which we can always integrate (12). Mr. Wilczynski has considered a class of surfaces for which the invariants $I=B_u/4BA^{\frac{1}{3}}$, and $J=A_v/4AB^{\frac{1}{3}}$, where $A=a'b^2$ and $B=a'^2b$, vanish identically, and he has shown that these surfaces are self-projective.§ For such surfaces we may assume without loss of generality that $a'=b=1$. Then equation (12) becomes $p'+2p^3-2=0$, where $p=u'$, and its first integral is

$$\log c\sqrt[6]{(p-1)^2/(p^2+p+1)}-(\sqrt[3]{3}/3) \text{ arc tan } (2p+1)/\sqrt{3}=-2v.$$

3. *The Definition of Torsal Curves and Their Relation to Union Curves.*

There are two well-known fundamental properties of congruences. First, the lines of a congruence are the common tangents of two surfaces, or more precisely, they are the double tangents of a surface with two sheets, the focal surface. Second, if the two sheets of the focal surface do not coincide point for point, the lines of the congruence may be assembled into two one-parameter families of developables. We shall determine the curves on S_y such that the ruled surface composed of the lines of L, corresponding to the points of these curves shall be developables. These curves we shall call *torsal curves.* There

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* Darboux, *Leçons,* Vol. III, Chap. 5.

† R. Liouville, "Sur une classe d'équations différentielles," *Comptes Rendus,* Vol. CV (1887), p. 1062.

‡ A. Guldberg, "On Geodesic Lines on Special Surfaces," *Nyt Tydsskrift. Math.,* Vol. VI (1895). (See *Jahrbuch,* "Ueber die Fortschritte der Mathematik.")

§ E. J. Wilczynski, "On a Certain Class of Self-Projective Surfaces," *Transactions of the American Mathematical Society,* Vol. XLV (1913), pp. 421–443.

will be two of these curves passing through each non-special point of the surface, one from each family. We shall assume, therefore, that the two sheets of the focal surface are distinct, that is, that it will be impossible to find two functions $w_1(u, v)$ and $w_2(u, v)$ such that

$$w_1(u, v)\xi^{(k)}+w_2(u, v)\eta^{(k)}=0 \qquad (k=1, 2, 3, 4),$$

where $\xi^{(k)}$ and $\eta^{(k)}$ are the coordinates of points on S_ξ and S_η, the two sheets of the focal surface.*

The equations of l_y, as given in (9), are satisfied by the coordinates $(0, -p_2, p_1, 1)$ so that the point P_t, where

$$t=-p_2z+p_1\rho+\sigma \tag{17}$$

is a point of l_y. If we allow u and v to take on the infinitesimal increments du and dv, the point P_y moves to the point $P_{y+y_u du+y_v d_v}$ and the point P_t to $P_{t+t_u du+t_v d_v}$ where

$$t_u=Py+Qz+R\rho+S\sigma, \quad t_v=P'y+Q'z+R'\rho+S'\sigma, \tag{18}$$

and

$$\left.\begin{aligned} P&=fp_2+2bg-f_v, & P'&=-gp_1+2a'f-g_u,\\ Q&=4a'b-p_{2u}, & Q'&=-2a'p_1-g-2a'_u-p_{2v},\\ R&=2bp_2-f-2b_v+p_{1u}, & R'&=4a'b+p_{1v},\\ S&=p_1, & S'&=-p_2.\end{aligned}\right\} \tag{19}$$

An arbitrary point P_x on the line $l_{y+y_u du+y_v d_v}$ will be represented by

$$\lambda(y+y_u du+y_v dv)+\mu(t+t_u du+t_v dv),$$

and its coordinates referred to T will be

$$\left.\begin{aligned} x_1&=\lambda+\mu(Pdu+P'dv), & x_2&=\lambda du+\mu(-p_2+Qdu+Q'dv),\\ x_3&=\lambda dv+\mu(p_1+Rdu+R'dv), & x_4&=\mu(1+Sdu+S'dv).\end{aligned}\right\} \tag{20}$$

In order that P_x may also be a point of l_y, its coordinates must satisfy (9), that is,

$$\left.\begin{aligned} \lambda[A_2du+A_3dv]+\mu[(A_2Q+A_3R+A_4S)du+(A_2Q'+A_3R'+A_4S')dv]=0,\\ \lambda[B_2du+B_3dv]+\mu[(B_2Q+B_3R+B_4S)du+(B_2Q'+B_3R'+B_4S')dv]=0,\end{aligned}\right\} \tag{21}$$

whence

$$\begin{vmatrix} A_2du+A_3dv & (A_2Q+A_3R+A_4S)du+(A_2Q'+A_3R'+A_4S')dv\\ B_2du+B_3dv & (B_2Q+B_3R+B_4S)du+(B_2Q'+B_3R'+B_4S')dv\end{vmatrix}=0. \tag{22}$$

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* E. J. Wilczynski, "Sur la theorie general des congruences," Bruxelles, 1911.

If we let

$$L=2bp_2-f-p_1^2-2b_v+p_{1u}, \quad 2M=p_{1v}+p_{2u}, \quad N=2a'p_1+g+p_2^2+2a'_u+p_{2v}, \tag{23}$$

then (22) becomes

$$Ldu^2+2Mdudv+Ndv^2=0; \tag{24}$$

this is the quadratic equation of the tangents to the torsal curves. If these curves form a conjugate system, there exists a function θ, such that $p_1=-\theta_u$ and $p_2=\theta_v$. When L is the directrix congruence of the second kind, the torsal curves are the directrix curves.* If L is the congruence of normals, they are the lines of curvature.

The question naturally arises whether some or all of the union curves might be plane. We shall show that this can happen if, and only if, they are at the same time torsal curves.

Every curve in three-dimensional space is characterized by a linear differential equation of the fourth order of the form,†

$$q_0y^{(\mathrm{IV})}+4q_1y'''+6q_2y''+4q_3y'+q_4y=0, \tag{25}$$

where $y^{(i)}=d^iy/dt^i$ $(i=1, 2, 3, 4)$, and $q_0, \ldots, q_4$ are in general functions of t. We may regard the solutions of (25) as the homogeneous coordinates of a point in space. As t varies this point describes a curve. Since

$$\eta_i=\sum_{k=1}^{4} c_{ik}y_k \qquad (i=1, 2, 3, 4)$$

is the most general solution of (25), we may say the differential equation defines a set of projectively equivalent curves in three-dimensional space. These curves will be plane if, and only if, $q_0=0$. We shall indicate the method for computing the fourth order differential equation which characterizes the union curves and actually calculate the value of q_0. From equations (1) and (6), and those obtained from them by partial differentiation, and from the value of u'' given by (12), we find the following formulae in which v is chosen as independent variable:

$$\left.\begin{aligned} y' &=u'z+\rho, & y'' &=a_1y+a_2z+a_3\rho+a_4\sigma, \\ y''' &=b_1y+b_2z+b_3\rho+b_4\sigma, & y^{(\mathrm{IV})} &=c_1y+c_2z+c_3\rho+c_4\sigma, \end{aligned}\right\} \tag{26}$$

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* Second Memoir, p. 116.

† Proj. Diff. Geom., Chap. 2.

where

$$\left.\begin{aligned}
&a_1=-fu'^2-g, \quad a_2=-2bu'^3-2p_1u'^2-2p_2u', \quad a_3=-2bu'^2, \quad a_4=2u',\\
&b_1=\ 6bfu'^4+(6fp_1-f_u)u'^3+3(2bg+2fp_2-f_v)u'^2-3g_uu'-g_v,\\
&b_2=-fu'^3+12a'bu'^2,\\
&b_3=\ 12b^2u'^4+(12bp_1-2b_u)u'^3+3(4bp_2-f-2b_v)u'^2-g,\\
&b_4=-8bu'^3-6p_1u'^2-6p_2u'+4a',\\
&c_1=[(2bg-f_v)b_4-fb_2+b_{1u}]u'-gb_3+(2a'f-g_u)b_4+b_{1v},\\
&c_2=(b_1+4a'bb_4+b_{2u})u'-2a'b_3-(g+2a'_u)b_4+b_{2v},\\
&c_3=[-2bb_2+b_{3u}-(f+2b_v)b_4]u'+b_1+4a'bb_4+b_{3v},\\
&c_4=(b_3+b_{4u})u'+b_2+b_{4v},
\end{aligned}\right\} \tag{27}$$

where $b_{1u},\ldots\ldots, b_{4u}, b_{1v},\ldots\ldots, b_{4v}$ are found from the fifth to eighth equations of (27) by partial differentiation, and u'' is given by (12). The desired equation will be obtained by eliminating z, ρ, and σ between equations (26) which gives

$$\begin{vmatrix} y' & u' & 1 & 0 \\ y''-a_1y & a_2 & a_3 & a_4 \\ y'''-b_1y & b_2 & b_3 & b_4 \\ y^{(\mathrm{IV})}-c_1y & c_2 & c_3 & c_4 \end{vmatrix}=0. \tag{28}$$

Thus the coefficient of $y^{(\mathrm{IV})}$ is

$$q_0=-\begin{vmatrix} u' & 1 & 0 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix},$$

or, substituting from (27),

$$q_0=4u'^2(Lu'^2+2Mu'+N). \tag{29}$$

Since u' can not vanish, $q_0=0$ is equivalent to (24). Hence we have proved the following theorem:

A necessary and sufficient condition that the union curves be plane is that they shall also be torsal curves.

This necessitates two relations between p_1, p_2, a', b and their derivatives which are obtained by the substitution in (12) of each of the pairs of values of u' and u'' found from (23) and the equation obtained from it by differentiation. If we let

$$R^2=M^2-LN, \tag{30}$$

these relations are

$$\left.\begin{aligned} &2a'L^3+8bM^3+2M^2(L_u-2p_1L)+L^2(2p_1N+2p_2M+M_v+\tfrac{1}{2}N_u) \\ &\qquad\qquad -LM(2M_u+L_v)-LN(6bM-\tfrac{1}{2}L_u)=0, \\ &R(4bM^2+2bLMN-4p_1LM+2p_2L^2-LM_u+2ML_u-LL_v) \\ &\qquad\qquad +L^2R_v-LMR_u=0. \end{aligned}\right\} \quad (31)$$

The second of these is satisfied identically when $R=0$, that is, when the two families of torsal curves coincide. For this case (31) reduces to the single condition

$$2a'L^2+2bM^3-2p_1LM^2+2p_2L^2M-LMM_u+L_uM^2+L^2M_v-LL_vM=0, \quad (32)$$

for particular values of p_1 and p_2, one must always show that these conditions are not inconsistent with the integrability conditions (2). This caution applies also to all similar situations which occur in the following pages.

Since by definition the osculating plane of a union curve at the point P_y contains the line l_y, and since the osculating plane of a plane curve is the same for all points of the curve, it is obvious geometrically that plane union curves are torsal curves, and that the developables are the planes of the curves themselves. If every curve of the two-parameter family of union curves is plane, it is evident that the torsal curves must be indeterminate, since there are only a single infinity of them. In that case we must have

$$L=M=N=0, \quad (33)$$

and conditions (31) would be satisfied identically.

In case all the union curves are plane, the lines of L for a restricted region R of the surface have a point in common as a simple geometric argument will show. Consider that part of a union curve C lying in R. It follows from the theory of differential equations that for R sufficiently small, there exist union curves joining two arbitrary points of C to a third point of R not on C. We obtain in this way an infinite number of triples of points for which the corresponding lines of L intersect in pairs. Since these lines are not all in the same plane, they must all pass through the same point, the edge of regression of all of the developables of the congruence.

The theorem proved above is of especial interest as it includes as a particular instance the well-known theorem that a geodesic is plane when, and only when, it is a line of curvature.

4. *The Principle of Duality Applied to Union Curves.*

The integral surface S_y of (1) may be regarded as the envelope of its tangent planes. It will then be represented by the partial differential equations.*

$$Y_{uu}-2bY_v+(2b_v+f)Y=0, \quad Y_{vv}-2a'Y_u+(2a'_u+g)Y=0, \tag{34}$$

where the solutions Y_i $(i=1, 2, 3, 4)$ are proportional to the homogeneous coordinates of the plane p_y tangent to the surface S_y at the point P_y. If we let

$$\bar{a}'=-a', \quad \bar{b}=-b, \quad \bar{f}=2b_v+f, \quad \bar{g}=2a'_u+g, \tag{35}$$

we may replace (34) by

$$Y_{uu}+2\bar{b}Y_v+\bar{f}Y=0, \quad Y_{vv}+2\bar{a}'Y_u+\bar{g}Y=0, \tag{36}$$

which is called the system adjoined to (1). It is evident from (35) that each is the adjoint of the other. If the solutions Y_i $(i=1, 2, 3, 4)$ are regarded as the coordinates of a plane, systems (1) and (36) have the same integral surfaces. But if they are regarded as the coordinates of a point, every integral surface S_Y of (36) would be a dualistic transform of every integral surface S_y of (1). Since $a'\neq 0$ and $b\neq 0$, S_y is not identically self-dual, that is, there exists no dualistic transformation carrying the point P_y over into the plane p_y tangent at that point. If Y_i be interpreted as the coordinates of a point, then the equation of the union curves on S_Y is

$$u''-2bu'^3+2\bar{p}_1u'^2+2\bar{p}_2u'+2a'=0. \tag{37}$$

In order to pass to the dualistic interpretation, let us regard S_y as the locus of its points, and S_Y as the envelope of its tangent planes. To the congruence L of lines l_y passing through the points of S_y, will correspond a congruence L_Y of lines l_Y in the tangent planes of S_Y. To a curve as point locus on S_y will correspond a developable circumscribed about the surface S_Y. Just as the union curves on S_y have the property that the osculating plane at P_y determined by three consecutive points of the curve contains the line l_y, so the point, which for symmetry we shall call the *osculating point for* p_y, determined by three consecutive planes of the developable, lies on the line l_Y. As the line l_y is the intersection of the osculating planes of all of the union curves passing through P_y, so also l_Y is the locus of all the osculating points of the developables containing p_y. To a plane union curve would correspond a cone with the osculating point as vertex. The theorems developed in the preceding pages are capable of dualistic interpretation, and could be developed independently by analysis dualistic to that employed above. The values of $\bar{p}_1$ and p_2

* First Memoir, p. 257.

in (37) evidently depend upon the choice of L_Y. As instance of projectively related congruences of this character one may cite the directrix congruences of the first and second kind.

5. *The Ruled Surface of the Congruence L along a Union Curve.*

We shall determine the differential equations which characterize the ruled surface generated by the line l_y as P_y moves along one of the union curves. The coordinates of a point t of the line l_y given in (17), and of y, must satisfy differential equations of the form *

$$n_{11}y''+p_{11}y'+p_{12}t'+q_{11}y+q_{12}t=0, \quad n_{22}t''+p_{21}y'+p_{22}t'+q_{21}y+q_{22}t=0, \tag{38}$$

where $y'=dy/dv$, $y''=d^2y/dv^2$ and so on. By means of (1), (12), (17), (18) and (19) we find

$$\left.\begin{aligned} &y=y, \quad y'=u'z+\rho, \quad y''=-(u'^2f+g)y+(u''-2a')z-2bu'\rho-2u'\sigma,\\ &t=-p_2z+p_1\rho+\sigma,\\ &t'=(u'P+P')y+(u'Q+Q')z+(u'R+R')\rho+(u'S+S')\sigma,\\ &t''=T_1y+T_2z+T_3\rho+T_4\sigma, \end{aligned}\right\} \tag{39}$$

where $T_1, \ldots, T_4$ are somewhat lengthy expressions in a', b, their partial derivatives and the powers of u' up to the third. By eliminating z, ρ and σ by means of the second, third, fourth and fifth of equations (39), and then from the second, fourth, fifth and sixth, there result two equations of type (38) where $n_{11}=n_{22}=-(Lu'^2+2Mu'+N)$. Since the ruled surface is a developable if, and only if, $n_{11}=n_{22}=0$, the results of § 3 are again emphasized.

The four pairs of independent solutions of (38), y_i, t_i, $(i=1,2,3,4)$ may be taken as the homogeneous coordinates of two points P_y and P_t. As the line l_y generates the ruled surface, these points describe curves C_y and C_t. The curve C_y is asymptotic for that surface provided that $p_{12}=0$.† The calculation of the coefficients gives

$$p_{12}=4u'(p_1u'+p_2). \tag{40}$$

Since $u'\neq 0$, $p_{12}=0$ when, and only when, $u'=-p_2/p_1$. This value of u' must satisfy (12) which imposes the following restriction

$$p_1p_2(p_{2u}+p_{1v})-p_2^2(p_{1u}+2bp_2)-p_1^2(p_{2v}+2a'p_1)=0. \tag{41}$$

That this condition may be satisfied is apparent. Let L be the congruence of directrices of the second kind, and let a' and b be functions of v alone, and of

* Proj. Diff. Geom., p. 126.

† Proj. Diff. Geom., p. 142.

u alone, respectively. Then $p_1 = p_2 = 0$. Whenever p_1 and p_2 are such that (41) is satisfied, there exists a one-parameter family of union curves having the property that they are at the same time asymptotic curves of the ruled surface generated by the lines of L along those curves.

6. *Some Remarks on a Problem in the Calculus of Variations.*

One of the most interesting properties of geodesics is that they appear in the problem of determining the lines of shortest length on a surface. Because so much of the theory of geodesics has turned out to be capable of generalization to the theory of union curves, the question naturally arises whether there exists a function $F(u, v, u')$ such that the integral $\int F(u, v, u')dv$ assumes a minimum value along a union curve, and such that the integral is invariant under the transformation $\bar{u} = \alpha(u)$, $\bar{v} = \beta(v)$. It is possible to find an infinity of functions satisfying the first of these conditions. The latter condition is essential in order to obtain an integral which may have an intrinsic projective significance. Investigation of this question has yielded thus far only negative results.